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Molien function for duality

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Abstract: The Molien function counts the number of independent group invariants of a representation. For chiral superfields, it is invariant under duality by construction. We illustrate how it calculates the spectrum of supersymmetric gauge theories.

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1. Introduction

It is quite remarkable that certain four dimensional gauge theories can be solved exactly. The examples that have been solved so far ([1] for a review) are quite special: they have lots of symmetries.

A generic theory does not have so many symmetries, so here I introduce a tool which I hope will be useful to the study of more general theories. To be concrete, I will consider only supersymmetric theories.

2. The Molien function

Consider a supersymmetric gauge theory with chiral superfields transforming as a representation R of a group G. I make no restrictions on R: it can be reducible, and also contain singlets (mesons). Similarly, G can be a product of groups, or it can be the identity, for a confining theory.

The Molien generating function for the representation R is

$$M(z) = \sum_{k=0}^{\infty} c_k z^k \, ,$$

where c_k is the number of independent group invariant polynomials of order k. It is a holomorphic function.

It turns out that there is a nice way to write down $M(z)$ (see [2] p. 204 for an easy proof):

$$M(z) = \int \frac{d\mu(g)}{\det(1 - zR(g))} .$$

The idea of the proof is that one can diagonalize the unitary representation R for any fixed group element g; then integration ($\int d\mu(g)$) over the whole group picks out only the singlets in the tensor products $R^{\otimes k}$.

The function M can be evaluated more explicitly (see the nice paper by Forger [3] for much useful and readable complementary details)

$$M(z) = \frac{1}{|W|} \int \cdots \int \frac{dw_1}{2\pi i w_1} \cdots \frac{dw_l}{2\pi i w_l} \frac{\Pi_\alpha \left(1 - w^{h(\alpha)}\right)}{\Pi_\lambda \left(1 - z w^{h(\lambda)}\right)} .$$

1. $|W|$ is the number of elements in the Weyl group
2. l is the rank of the group;
3. the products are over all the roots α of the group and over all the weights λ of the representation R;
4. and finally, the concise notation $w^{h(\alpha)}$ means $w_1^{h(\alpha_1)} \cdots w_l^{h(\alpha_l)}$, where $h(\alpha_i)$ is the eigenvalue of the root α under the Cartan generator H_i.

Another representation of the Molien function coefficients is given in terms of an index:

$$c_k = \frac{1}{|W|} \sum_{\tilde{\lambda}} i\left(\tilde{\lambda}\right) m_k\left(\tilde{\lambda}\right),$$

with the $\tilde{\lambda}$ denoting the extended weights and m_k the k-extended multiplicities. This terminology is defined in [3]. According to [3], the index might be more efficient for explicit calculations.

3. Duality[1]

For a $\mathcal{N} = 1$ supersymmetric gauge theory with a vanishing superpotential, the Molien function calculated for the gauge group of the theory encodes much information about the low-energy spectrum. It calculates how many gauge invariant independent chiral (holomorphic) operators there are of a given degree in the number of elementary fields. In other words, it contains much about the structure of the chiral ring.

[1] I wish to thank O. Aharony and A. Schwimmer for inquiries that led to a clarification of this section.

Consider now a dual "magnetic" description to this theory. Since by assumption it has the same low-energy spectrum, there must be a way, expected to be complicated (i.e. **not** a Molien function!), to calculate the Molien function of the "electric" theory in terms of the data of the magnetic theory. In this sense the Molien function is duality invariant.

Since it is a holomorphic function, one would hope that the powerful tools of complex analysis can be useful to study its properties. This, however, is highly speculative, as well as the remaining of this paragraph. Even if it is not enough to fully characterize a supersymmetric conformally invariant gauge theory, the Molien function could be "the" characteristic function of $\mathcal{N}=1$ duality.[2] It is definitely interesting because it does not rely on global symmetries. For a generic theory, global symmetries are small and the constraints one can get from them have a limited power: it is well known that satisfying the 't Hooft anomaly matching conditions is not enough. In string theory, there are no global symmetries anyway. Perhaps dynamical properties of the low-energy theory can be inferred from the chiral spectrum. This would come about by making the following statement more precise: the Molien function of a confining theory is simple, while the Molien function of a gauge theory which is not asymptotically free is complicated (it has high order syzygies among its invariants.)

If the theory has a non-zero superpotential, extra constraints are introduced among the invariants. The definition of the Molien function stays the same (namely $M(z)=\sum_{k=0}^{\infty} c_k z^k$ where c_k is the number of independent group invariant polynomials of order k), but the integral representation ($\int \frac{d\mu(g)}{\det(1-zR(g))}$) should be generalized to include the effect of the superpotential (I don't know how to write it down).

4. Generalized Molien Function

It would be nice to have an explicit way to construct the Molien function of the electric from the magnetic theory. One might hope that there is a generalization of the Molien function, $\tilde{M}$, which is such that calculating $\tilde{M}$ in the electric and in the magnetic theory would give the same result. I do not know if this is possible.

As a step in this direction, it is convenient to define a generalized Molien function, still assuming that the superpotential is zero, by choosing a global $U(1)$ charge, under which the elementary fields in irreducible representation matrices R_i transform with charges q_i, $i=1,\ldots,n$ (which can all be taken to be integers by suitable rescaling).

[2]However, there is no claim of uniqueness here: with the meaning of duality invariance above, any formal function of variables z_α (one variable for each chiral invariant operator O_α) will be duality invariant, as long as the constraints among the operators are suitably implemented by constraints among the z_α, a feat that the Molien function accomplishes naturally. See section 4.

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With this, the generalized Molien function

$$M_{\{q_i\}} = \int d\mu(g)/\det \begin{pmatrix} 1 - z^{q_1}R_1(g)) & & & \\ & 1 - z^{q_2}R_2(g)) & & \\ & & \ddots & \\ & & & 1 - z^{q_n}R_n(g)) \end{pmatrix}$$

has the property that the invariant operators are counted with the same power of k as coefficients of z^k in the electric and in the magnetic theories. The coefficients will still disagree of course because the constraints from the superpotentials have not been included.

5. Illustration

Aside from the duality application, the Molien function provides a technique to grind out the spectrum of a theory, along with plethysms, branching rules and other counting arguments. I will illustrate some of the uses with the simplest examples. Start with $\mathcal{N} = 1$ supersymmetric $SU(2)$ gauge theory with one flavor of fundamentals Q_i (two doublets) [1].

Evaluating $M(z)$ with the integral representation readily gives

$$M = \frac{1}{1-z^2} = 1 + z^2 + z^4 + \cdots .$$

This generating function is characteristic of a freely generated ring with one invariant: there's one polynomial of order 2, namely Q_1Q_2, and one of order 4, $(Q_1Q_2)^2$, and so on.

With 4 doublets,

$$M(z) = \frac{1-z^4}{(1-z^2)^6} = 1 + 6z^2 + 20z^4 + 50z^6 + \cdots .$$

The coefficient 6 indicates that the ring is generated by the invariants $V_{ij} = Q_iQ_j$. At order z^4, we learn that the V_{ij} are not independent, but there is one constraint among them, the famed pf V $= \Lambda^4$. Studying the following coefficients shows that there are no more constraints.

With 6 doublets,

$$M(z) = \frac{1 + 6z^2 + 6z^4 + z^6}{(1-z^2)^9} = 1 + 15z^2 + (120-15)z^4 + (680-189-1)z^6 + \cdots .$$

This is already more complicated. There are 15 invariants V_{ij}, and 15 constraints (syzygies) $\epsilon^{ijklmn}V_{kl}V_{mn}$, but there are constraints amongst the constraints and so on.

More generally, with d doublets,

$$M_d = \sum_{k=0}^{\infty} \dim \left(\begin{smallmatrix} \square & \square & \cdots & \square \\ \square & \square & & \square \end{smallmatrix} \right) z^k ,$$

where the tensor under the $SU(2d)$ symmetry has k horizontal boxes.

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6. New example

As the rank of the group increases, the formulas for the Molien function become rapidly cumbersome to evaluate. For the integral representation, one is faced with high order poles to be evaluated by the residue theorem. A trick is to settle for less than the full generating function, and get only the first few c_k: one takes derivatives with respect to z, and then set $z = 0$, before evaluating the residues at the w_i, for which the poles are now automatically all at $w_i = 0$. To go beyond that, perhaps one could reexpress these integrals using Littlewood's Schur functions, or use the index formula of [3]. Another possibility to evaluate the Molien function more effectively is to use the MacMahon algorithm [4].[3]

Here for simplicity, I will only calculate the spectrum of the supersymmetric $SU(2)$ gauge theories with one matter field in the 4-dimensional representation S and $2k$ doublet fields Q_i.

1. $k = 0$. This theory was studied in [5]. There is just one invariant, quartic, so $M = 1/(1 - z^4)$.

   When $k > 0$, the theories are not asymptotically free. That does not make them uninteresting, because they can still be the free duals of strongly coupled theories.

2. $k = 1$
$$M = \frac{1 - z^2 + 5z^4 - z^6 + z^8}{(1 - z^4)^3(1 - z^2)^2} .$$
   At this stage, we see the invariants Q^2, SQ^3, S^2Q^2, S^3Q^3 and S^4. The invariants SQ^3, S^2Q^2, S^3Q^3 are fully symmetric in their flavor indices. They generate the full ring, but they are not independent. Checking this result for $k > 1$, we see that these invariants still form a full set, but there are more constraints.

3. $k = 2$
$$M = \frac{1 + 2z^2 + 28z^4 + 23z^6 + 73z^8 + 23z^{10} + 28z^{12} + 2z^{14} + z^{16}}{(1 - z^4)^5(1 - z^2)^4} .$$

4. $k = 3$
$$M = \frac{1 + 9z^2 + 101z^4 + 319z^6 + 1020z^8 + 1475z^{10}}{(1 - z^4)^7(1 - z^2)^6} + \\ + \frac{2091z^{12} + 1475z^{14} + 1020z^{16} + 319z^{18} + 101z^{20} + 9z^{22} + z^{24}}{(1 - z^4)^7(1 - z^2)^6} .$$

[3] I thank C. Cummins for pointing out the usefulness of [4] in this respect.

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